

0.1 POLYNOMIALS AND RATIONAL FUNCTIONS



Preliminaries

The Real Number System and Inequalities

CHAPTER

0

الطبيعية Natural numbers = $N = \{1, 2, 3, 4, \dots\}$

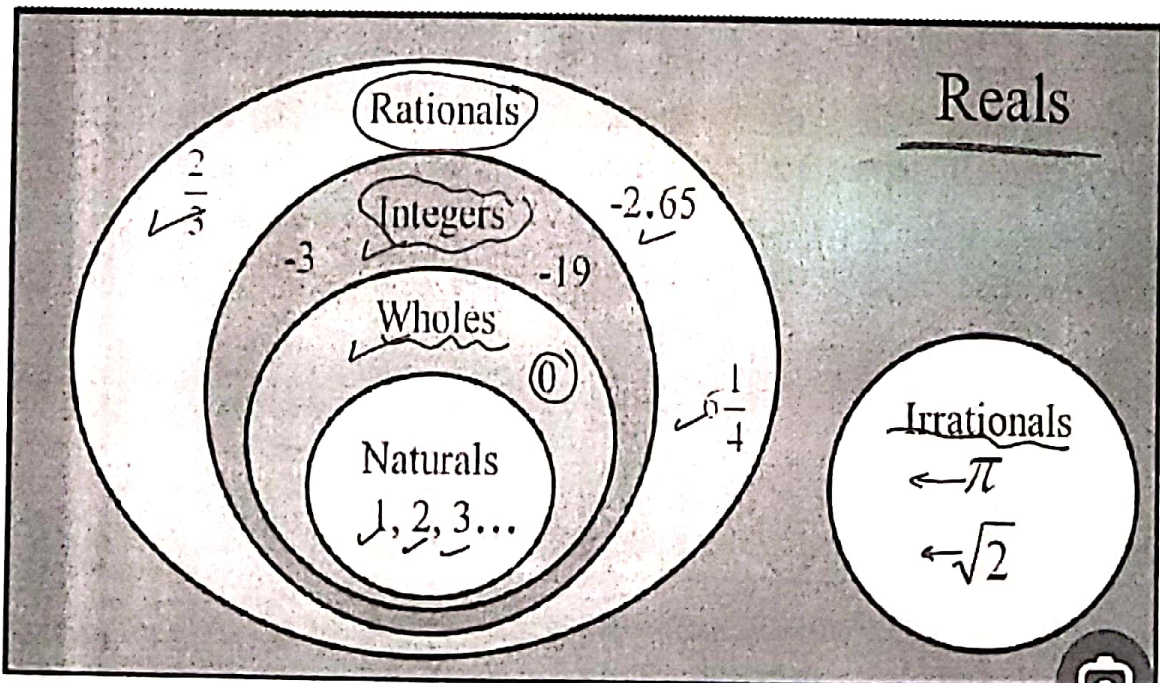
الكلية whole numbers = $W = \{0, 1, 2, 3, \dots\}$

الاصوية Integers numbers = $Z = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$

2 50%, 0.75, $\frac{1}{2}$, $\frac{2}{3}$, $6\frac{4}{6}$ = Rational numbers ① النسبية

غير النسبية Irrational numbers = $\{2.25, e, \pi, \sqrt{5}, \sqrt{2}\}$

Real numbers = R



الفترات Intervals

غير محدودة

محدودة

...unbounded

bounded

$x > 7$ غير محدود $(7, \infty)$

$x < 7$ غير محدود $(-\infty, 7)$

open interval (a, b)

closed interval $[a, b]$

$x < 2$ غير محدود $(-\infty, 2)$

$x \leq -5$ غير محدود $(-\infty, -5]$

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$



$$5 < x \leq 7$$

$$(5, 7]$$

فترة نصف مفتوحة

أو نصف مغلقة

THEOREM 1.1

If a and b are real numbers and $a < b$, then

- (i) For any real number c , $a + c < b + c$
- (ii) For real numbers c and d , if $c < d$, then $a + c < b + d$.
- (iii) For any real number $c > 0$, $a \cdot c < b \cdot c$
- (iv) For any real number $c \leq 0$, $a \cdot c > b \cdot c$.

ضرب المتباينة

$$\begin{array}{l} -x < 2 \\ (-1) x < (-1) 2 \\ x > -2 \end{array}$$

$$\begin{array}{l} -2x > 6 \\ \frac{-2x}{-2} > \frac{6}{-2} \\ x < -3 \end{array}$$

القسمة على عدد
سالب تعكس العلامة

REMARK 1.1

We need the properties given in Theorem 1.1 to solve inequalities.

EXAMPLE 1.1 Solving a Linear InequalitySolve the linear inequality $2x + 5 < 13$.

(X)

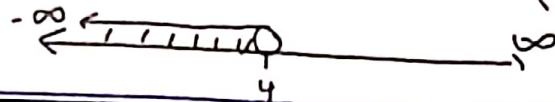
$$2x + 5 < 13$$

$$2x < 13 - 5 \Rightarrow (2)x < 8$$

 $\div 2$

$$\frac{2x}{2} < \frac{8}{2}$$

$$x < 4 \Rightarrow S.I = (-\infty, 4)$$

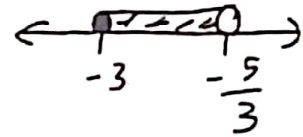
**EXAMPLE 1.2** Solving a Two-Sided InequalitySolve the two-sided inequality $6 < 1 - 3x \leq 10$.

$$6 < 1 - 3x \leq 10$$

$$6 - 1 < -3x \leq 10 - 1$$

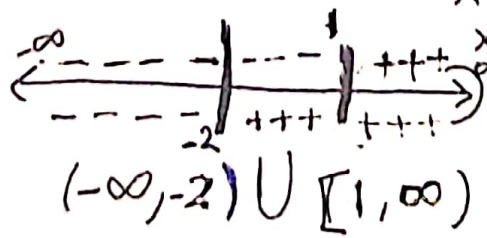
$$\frac{5}{-3} < \frac{-3x}{-3} \leq \frac{9}{-3} \quad \div (-3)$$

$$-\frac{5}{3} > x \geq -3 \Rightarrow [-3, -\frac{5}{3})$$

**EXAMPLE 1.3** Solving an Inequality Involving a FractionSolve the inequality $\frac{x-1}{x+2} \geq 0$.

$$x - 1 = 0$$

$$x = 1$$



اصفا الفاء

$$x + 2 = 0$$

$$x = -2$$

كسر
 ① اصفا البسط
 ② اصفا المقام
 ③ اصفا البسط
 ④ اصفا المقام

$$\frac{+}{+} = +$$

$$\frac{-}{-} = +$$

EXAMPLE 1.4 Solving a Quadratic Inequality

Solve the quadratic inequality

حل المتباينة التربيعية

$$x^2 + x - 6 \geq 0.$$

ترتيب

مميز

$$x^2 + x - 6 > 0$$

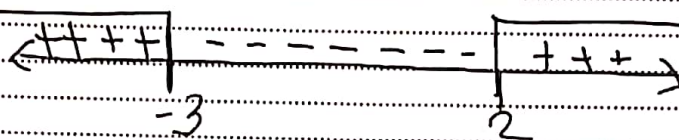
$$(x-2)(x+3) > 0$$

$$x-2=0$$

$$x+3=0$$

$$x=2$$

$$x=-3$$



$$= (-\infty, -3) \cup (2, \infty)$$

$$x^2 + 5x + 6 =$$

$$(x+2)(x+3)$$

العلامة بين العددين عكس علامة x^2

والطرفين مثل علامة x^2

0.1 POLYNOMIALS AND RATIONAL FUNCTIONS

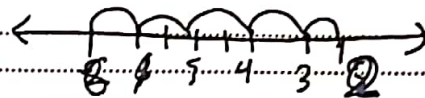
DEFINITION 1.1

$$|2| = 2, \quad |-2| = 2$$

The absolute value of a real number x is $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$$\otimes |a \cdot b| = |a| \cdot |b|, \quad |2 \cdot 3| = |2| \cdot |3| = 6$$

$$\otimes |a + b| \neq |a| + |b|,$$



$$\otimes |a + b| \leq |a| + |b|.$$

$$|6 - 2| = |4| = 4$$

NOTE THAT

Recall that for any real number $r > 0$, $|x| < r$ is equivalent to the following inequality not involving absolute values:

$$-r < x < r$$

EXAMPLE 1.5 Solving an Inequality Containing an Absolute Value

Solve the inequality

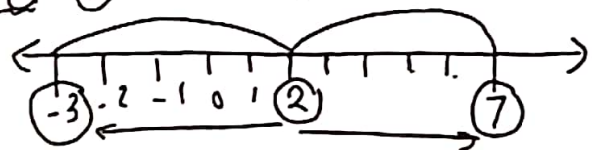
$$|x - 2| < 5$$

$$|x - 2| < 5$$

$$-5 < x - 2 < 5$$

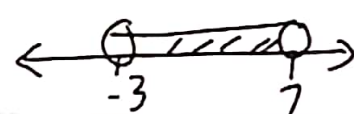
$$-5 + 2 < x < 5 + 2$$

$$-3 < x < 7$$



$$-3 < x < 7$$

$$(-3, 7)$$



EXAMPLE 1.6 Solving an Inequality with a Sum Inside an Absolute

Solve the inequality

$$|x + 4| \leq 7.$$

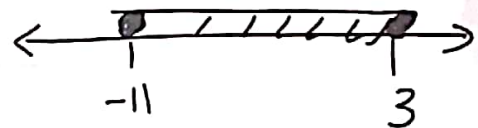
$$|x + 4| \leq 7$$

$$-7 \leq x + 4 \leq 7$$

$$\frac{-7-4}{-11} \leq x \leq \frac{7-4}{3}$$

$$-11 \leq x \leq 3$$

$$[-11, 3]$$



$$|x - 1| < 5$$

$$\phi = 2.5$$

ملاحظة

$$5$$

مرفوعة

EXAMPLE 1.7 An Alternative Method for Solving InequalitiesSolve the inequality $|x - 2| < 5$.

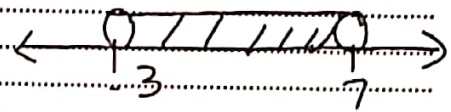
$$|x - 2| < 5$$

$$-5 < x - 2 < 5$$

$$-5 + 2 < x < 5 + 2$$

$$-3 < x < 7$$

$$S.I. = (-3, 7)$$

the distance between two points (x_1, y_1) and (x_2, y_2)

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**EXAMPLE 1.8** Using the Distance FormulaFind the distance between the points $(\overset{x_1}{1}, \overset{y_1}{2})$ and $(\overset{x_2}{3}, \overset{y_2}{4})$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(3 - 1)^2 + (4 - 2)^2}$$

$$= \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8}$$

○ Equations of Lines معادلات الخطوط

DEFINITION 1.2

For $x_1 \neq x_2$, the slope of the straight line through the points (x_1, y_1) and (x_2, y_2) is the number

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{فرق الـ } y \text{ (الارتفاع)}}{\text{فرق الـ } x \text{ (المسافة)}} \quad (1.5)$$

When $x_1 = x_2$ and $y_1 \neq y_2$, the line through (x_1, y_1) and (x_2, y_2) is vertical and the slope is undefined.

EXAMPLE 1.9 Finding the Slope of a Line

Find the slope of the line through the points $(x_1, y_1) = (4, 3)$ and $(x_2, y_2) = (2, 5)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{2 - 4} = \frac{2}{-2} = -1$$

$(2, 1), (2, 3)$ $m = \text{غير معرف}$ (لوازي محور y)
 $(2, 2), (3, 2)$ $m = 0$ (لوازي محور x)

EXAMPLE 1.11 Graphing a Line

If a line passes through the point $(x_1, y_1) = (2, 1)$ with slope $m = \frac{2}{3}$, find a second point on the line and then graph the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (3, \frac{5}{3})$$

$$\frac{y_2 - 1}{x_2 - 2} = \frac{2}{3}$$

ضع $x_2 = 3$

$$\frac{y_2 - 1}{3 - 2} = \frac{2}{3} \Rightarrow \frac{y_2 - 1}{1} = \frac{2}{3} \Rightarrow 3y_2 - 3 = 2$$

$$3y_2 = 5 \Rightarrow y_2 = \frac{5}{3}$$

المطلوب: النقطة

POINT-SLOPE FORM OF A LINE

$$y = m(x - x_0) + y_0$$

(1.7)

EXAMPLE 1.12 Finding the Equation of a Line Given Two Points

Find an equation of the line through the points (x_0, y_0) and (x_2, y_2) , and graph the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 1}{4 - 3} = \frac{-2}{1} = -2$$

نوصيه بـ ١٠٨ =

$$y = m(x - x_0) + y_0$$

$$y = -2(x - 3) + 1$$

مستقيمات

perpendicular



$$m_1 \cdot m_2 = -1$$

مستقيمات متعامدتان = -1

$$m_1 = \frac{2}{3}, m_2 = -\frac{3}{2}$$

٠٥٩٠٦٤٧١٧١

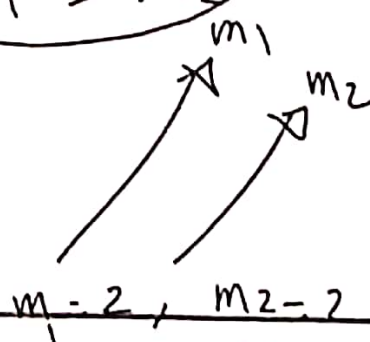
LINES

مستويات

parallel

مستويات متوازية

$$m_1 = m_2$$



$$m_1 = 2, m_2 = 2$$

١/ ابو عمر

EXAMPLE 1.13 Finding the Equation of a Parallel Line

Find an equation of the line parallel to $y = 3x - 2$ and through the point $(-1, 3)$.

لوازي

ميل الخط = 3

ميل المطلوب = 3

 (x_0, y_0)
 $(-1, 3)$

$$y = m(x - x_0) + y_0$$

$$y = 3(x + 1) + 3$$

EXERCISES 0.1

امثلية حل

In exercises 1-10, solve the inequality.

1. $3x + 2 < 8$

$3x < 8 - 2$

$3x < 6$

$x < 2$

$S.I = (-\infty, 2)$

3. $1 \leq 2 - 3x < 6$

$1 - 2 \leq -3x < 6 - 2$

$-1 \leq -3x < 4$

$-\frac{1}{3} \leq x < -\frac{4}{3}$

$-\frac{1}{3} \geq x > -\frac{4}{3} \Rightarrow (-\frac{4}{3}, -\frac{1}{3}]$

6. $\frac{2x+1}{x+2} \leq 0$

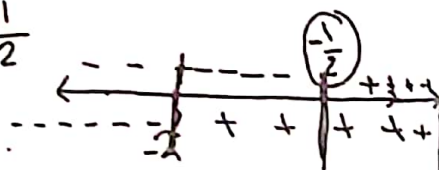
$2x+1 = 0$

$2x = -1$

$x = -\frac{1}{2}$

$x+2 = 0$

$x = -2$



$= (-2, -\frac{1}{2}]$

8. $x^2 - 5x - 6 < 0$

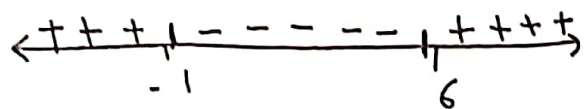
$(x+1)(x-6) < 0$

$x+1 = 0$

$x = -1$

$x-6 = 0$

$x = 6$



$S.I = (-1, 6)$

10. $|2x + 1| < 4$

$$-4 < 2x + 1 < 4$$

$$-4 - 1 < 2x < 4 - 1 \quad \left(-\frac{5}{2}, \frac{3}{2}\right)$$

$$-\frac{5}{2} < \frac{2x}{2} < \frac{3}{2} \Rightarrow -\frac{5}{2} < x < \frac{3}{2}$$

In exercises 15–18, find (a) the distance between the points, (b) the slope of the line through the given points, and (c) an equation of the line through the points.

15. $(1, 2), (3, 6)$

(a) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
 $d = \sqrt{(3-1)^2 + (6-2)^2} = \sqrt{2^2 + 4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

(b) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 2}{3 - 1} = \frac{4}{2} = 2$

(c) $y = m(x - x_1) + y_1 \Rightarrow y = 2(x - 1) + 2$

In exercises 23–28, determine if the lines are parallel, perpendicular, or neither.

23. $y = 3(x - 1) + 2$ and $y = 3(x + 4) - 1$

$$y = 3x - 3 + 2$$

$$y = 3x - 1$$

$$m_1 = 3$$

$$y = 3x + 12 - 1$$

$$y = 3x + 11$$

$$m_2 = 3$$

$$m_1 = m_2$$

parallel

$m_1 = \frac{2}{3}$
 $m_2 = \frac{-3}{2}$

EXERCISES 0.1

In exercises 29-32, find an equation of a line through the given point and (a) parallel to and (b) perpendicular to the given line.

29. $y = 2(x + 1) - 2$ at $(2, 1)$

$$y = 2x + 2 - 2 \Rightarrow y = 2x$$

ميل المستقيم المعطى = 2

(a) Parallel $\Rightarrow m = 2$, $(x_0, y_0) = (2, 1)$

$$y = m(x - x_0) + y_0$$

$$y = 2(x - 2) + 1$$

(b) perpendicular $m = -\frac{1}{2}$

$(x_0, y_0) = (2, 1)$, $m = -\frac{1}{2}$

$$y = m(x - x_0) + y_0$$

$$y = -\frac{1}{2}(x - 2) + 1$$

In exercises 65-72, factor and/or use the quadratic formula to find all zeros of the given function.

65. $f(x) = x^2 - 4x + 3$

عزيم ٣، ٢، ١
 $f(x) = 0 \Rightarrow x^2 - 4x + 3 = 0$
 $(x-3)(x-1) = 0$
 $x-3=0 \quad | \quad x-1=0$
 $x=3 \quad | \quad x=1$

67. $f(x) = x^2 - 4x + 2 = 0$
 $a=1 \quad b=-4 \quad c=2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2} = \frac{4 \pm \sqrt{16-8}}{2}$$

$$x = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

69. $f(x) = x^3 - 3x^2 + 2x$

$$f(x) = 0 \Rightarrow x^3 - 3x^2 + 2x = 0$$

$$x(x^2 - 3x + 2) = 0$$

$$x(x-2)(x-1) = 0$$

$$x=0 \quad | \quad x-2=0 \quad | \quad x-1=0$$

$$x=0 \quad | \quad x=2 \quad | \quad x=1$$

الحال

In exercises 43-48, find the domain of the function.

43. $f(x) = \sqrt{x+2}$
ماقت الكيز $\leq$

$$x+2 \geq 0$$

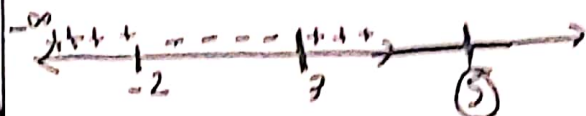
$$x \geq -2$$

$$D_f = [-2, \infty)$$

45. $f(x) = \frac{\sqrt{x^2 - x - 6}}{x - 5}$

القام $x \neq 5$

$$\begin{aligned} x^2 - x - 6 &\geq 0 \\ (x+2)(x-3) &\geq 0 \\ x+2=0 & \quad x-3=0 \\ x=-2 & \quad x=3 \end{aligned}$$



$$D_f = (-\infty, -2) \cup (3, 5) \cup (5, \infty)$$

47. $f(x) = \frac{4}{x^2 - 1}$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$



$$D_f = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

$$D_f = \mathbb{R} - \{-1, 1\}$$

48. $f(x) = \frac{4x}{x^2 + 2x - 6}$

ضع القام $x^2 + 2x - 6 = 0$
 $a=1 \quad b=2 \quad c=-6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-6)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{28}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{7}}{2} \Rightarrow \begin{cases} -1 + \sqrt{7} \\ -1 - \sqrt{7} \end{cases}$$

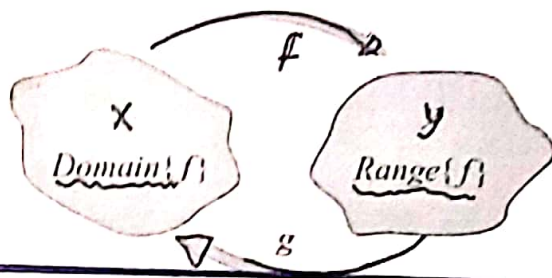
$$D_f = \mathbb{R} - \{-1 + \sqrt{7}, -1 - \sqrt{7}\}$$



0.3

INVERSE FUNCTIONS

$$g = f^{-1}$$



$$D_f = R_g$$

$$R_f = D_g$$

$$y = x^3 \Rightarrow x = \sqrt[3]{y} =$$

EXAMPLE 3.1 Two Functions That Reverse the Action of Each Other

If $f(x) = x^3$ and $g(x) = x^{1/3}$, show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

for all x .

$$f(g(x)) = (x^{1/3})^3 = x$$

$$g(f(x)) = (x^3)^{1/3} = x$$

DEFINITION 3.1

Assume that f and g have domains A and B , respectively, and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$. If

$$f(g(x)) = x, \quad \text{for all } x \in B, \quad \text{and}$$

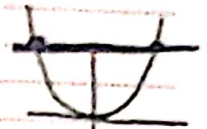
$$g(f(x)) = x, \quad \text{for all } x \in A,$$

we say that g is the inverse of f , written $g = f^{-1}$. Equivalently, f is the inverse of g , $f = g^{-1}$.

REMARK 3.1

$$f^{-1} \neq \frac{1}{f}$$

Notice that $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$. We write $\frac{1}{f(x)} = [f(x)]^{-1}$.

EXAMPLE 3.2 A Function with No InverseShow that $f(x) = x^2$ has no inverse on the interval $(-\infty, \infty)$.

$$f(4) = 16$$

$$f(-4) = 16$$

اي خط افقي يقطع

(x)

y

f has no inverse

تكون الدالة ليس لها معكوس

الرسمه من نقطتين

For $f(x) = x^2$ it is tempting to jump to the conclusion that $g(x) = \sqrt{x}$ is the inverse of $f(x)$ $x \geq 0$

شرط

$$f(g(x)) = (\sqrt{x})^2 = x$$

$$g(f(x)) = \sqrt{x^2} = |x| = x$$

$$f(g(x)) = g(f(x)) \Rightarrow f^{-1}(x) = \sqrt{x}$$

DEFINITION 3.2

الدالة تكون واحد لواحد اذا كانت لكل عنصر من المجال له عنصر واحد من المجال

A function f is called one-to-one when for every $y \in \text{Range}\{f\}$, there is exactly one $x \in \text{Domain}\{f\}$ for which $y = f(x)$.**REMARK 3.2**الدالة واحد لواحد اذا كانت one to one.A function $f(x)$ is one-to-one if and only if the equality $f(a) = f(b)$ implies $a = b$.

$$f(x) = x^7 - 7$$

$$f(x_1) = f(x_2)$$

$$x_1^7 - 7 = x_2^7 - 7$$

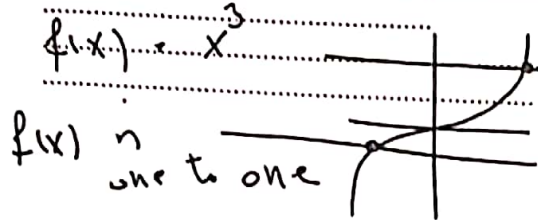
$$\Rightarrow x_1^7 = x_2^7$$

$$x_1 = x_2$$

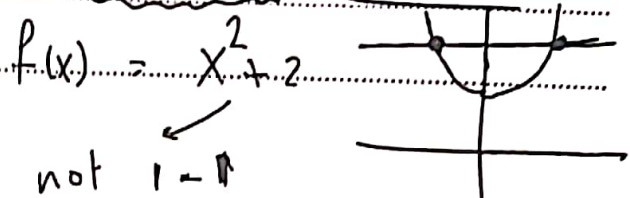
horizontal line test.

اختبار الخط الأفقي

function f is one-to-one if and only if every horizontal line intersects the graph in at most one point.



$f(x)$ has inverse
لها معكوس



$f(x)$ has no inverse
لا يوجد خط افقي يقطع الرسم في نقطتين
لها معكوس

THEOREM 3.1 الدالة لها معكوس إذا وفقط إذا كانت احادية

A function f has an inverse if and only if it is one-to-one.

one to one $\longrightarrow$ inverse

EXAMPLE 3.3 Finding an Inverse Function

Find the inverse of $f(x) = x^3 - 5$.

f is one to one

one to one

$$f(x_1) = f(x_2)$$

$$x_1^3 - 5 = x_2^3 - 5$$

$$x_1^3 = x_2^3$$

$$x_1 = x_2$$

$$y = x^3 - 5$$

$$x = y^3 - 5$$

$$x + 5 = y^3$$

$$\sqrt[3]{x+5} = \sqrt[3]{y^3}$$

$$y = \sqrt[3]{x+5} \Rightarrow f^{-1}(x) = \sqrt[3]{x+5}$$

لكن توجد معكوس دالة لا بد من ان تكون one to one

EXERCISES 0.3

In exercises 1-4, show that $f(g(x)) = x$ and $g(f(x)) = x$ for all x :

3. $f(x) = 2x^3 + 1$ or $g(x) = \sqrt[3]{\frac{x-1}{2}}$

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x-1}{2}}\right)^3 + 1 = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = x$$

ضع $g(x)$ مكان x في f

$$g(f(x)) = \sqrt[3]{\frac{2x^3 + 1 - 1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$$

ضع $f(x)$ بدل x في g

الـ x في g

4. $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1-2x}{x}$ ($x \neq 0, x \neq -2$)

$$f(g(x)) = \frac{1}{\frac{1-2x}{x} + 2} = \frac{1}{\frac{1-2x+2x}{x}} = \frac{1}{\frac{1}{x}} = x$$

$$g(f(x)) = \frac{1-2\left(\frac{1}{x+2}\right)}{\frac{1}{x+2}} = \frac{1 - \frac{2}{x+2}}{\frac{1}{x+2}} = \frac{\frac{x+2-2}{x+2}}{\frac{1}{x+2}} = \frac{\frac{x}{x+2}}{\frac{1}{x+2}} = x$$

$$g(f(x)) = \frac{1-2\left(\frac{1}{x+2}\right)}{\frac{1}{x+2}} = \frac{1 - \frac{2}{x+2}}{\frac{1}{x+2}} = \frac{\frac{x+2-2}{x+2}}{\frac{1}{x+2}} = \frac{\frac{x}{x+2}}{\frac{1}{x+2}} = x$$

Ⓜ In exercises 5-12, determine whether the function has an inverse (is one-to-one). If so, find the inverse and graph both the function and its inverse.

5. $f(x) = x^3 - 2$

① one to one

$f(x_1) = f(x_2)$

$x_1^3 - 2 = x_2^3 - 2$

$x_1^3 = x_2^3$

f is one to one

$y = x^3 - 2$

$x = y^2 - 2$

$x + 2 = y^2$

$\sqrt{x+2} = \sqrt{y^2}$

$y = \sqrt{x+2} \Rightarrow f^{-1}(x) = \sqrt{x+2}$

ميراث $y = f(x)$

يحل x من $y = f(x)$

ثم y

خذ

7. $f(x) = x^5 - 1$

$f(x_1) = f(x_2)$

$x_1^5 - 1 = x_2^5 - 1$

$x_1^5 = x_2^5$

$x_1 = x_2$

f is one to one

$y = x^5 - 1$

$x = y^5 - 1$

$x + 1 = y^5$

$\sqrt[5]{x+1} = \sqrt[5]{y^5}$

$y = \sqrt[5]{x+1} = (x+1)^{\frac{1}{5}} = f^{-1}$

9. $f(x) = x^4 + 2$

$f(x_1) = f(x_2)$

$x_1^4 + 2 = x_2^4 + 2$

$x_1^4 = x_2^4$

$x_1 = \pm x_2$

f is not one to one

ميراث

f has no inverse

In exercises 13–18, assume that the function has an inverse. Without solving for the inverse, find the indicated function values.

14. $f(x) = x^3 + 2x + 1$,

(a) $f^{-1}(1)$, (b) $f^{-1}(13)$

$$f(0) = 0^3 + 2(0) + 1 = 1 \Rightarrow f(0) = 1 \Rightarrow f^{-1}(1) = 0$$

$$f(2) = 2^3 + 2(2) + 1 = 8 + 4 + 1 = 13 \Rightarrow f(2) = 13 \Rightarrow f^{-1}(13) = 2$$

Exercises 37–46 involve inverse functions on restricted domains.

37. Show that $f(x) = x^2 (x \geq 0)$ and $g(x) = \sqrt{x} (x \geq 0)$ are inverse functions. Graph both functions.

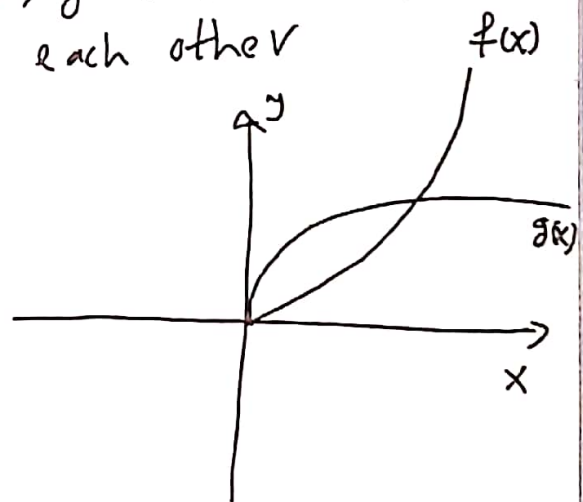
$$f(g(x)) = (\sqrt{x})^2 = x$$

$$g(f(x)) = \sqrt{x^2} = |x| = x \quad \left(\begin{array}{l} x \\ -x \\ x \end{array} \right)$$

$$= x$$

$$f(g(x)) = g(f(x))$$

f, g are inverse for each other





0.4

TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

DEFINITION 4.1

A function f is periodic of period T if

$$f(x + T) = f(x)$$

for all x such that x and $x + T$ are in the domain of f . The smallest such number $T > 0$ is called the fundamental period.

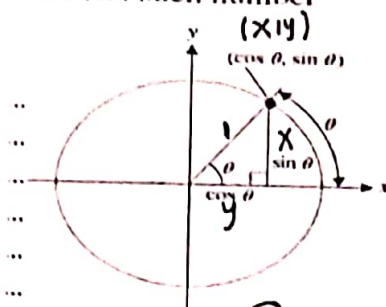
$$\sin(x + 2\pi) = \sin x$$

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t .

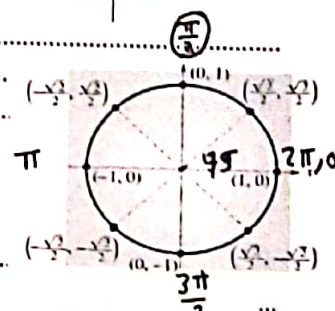
$\sin \theta = y$ $\cos \theta = x$

$$\tan \theta = \frac{y}{x}, \quad x \neq 0$$



$$x^2 + y^2 = 1 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1$$

$(0, 1)$
 $\sin \frac{\pi}{2} = 1$ $\cos \frac{\pi}{2} = 0$



From this definition, it follows that $\sin \theta$ and $\cos \theta$ are defined for all values of θ so that each has domain $-\infty < \theta < \infty$, while the range for each of these functions is the interval $[-1, 1]$.

المركبات الزاوية

Angle in degrees	0°	30°	45°	60°	90°	135°	180°	270°	360°
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

$30^\circ \times \frac{\pi}{180} = \frac{\pi}{6}$ $\frac{\pi}{2} \cdot \frac{180}{\pi} = \frac{180}{1} = 90$

* Conversions Between Degrees and Radians

- To convert degrees to radians, multiply degrees by $\frac{\pi \text{ rad}}{180^\circ}$
- To convert radians to degrees, multiply radians by $\frac{180^\circ}{\pi \text{ rad}}$

٠٥٩٠٦٤٧١٧١

ا/ ابو عمر

THEOREM 4.1

The functions $f(\theta) = \sin \theta$ and $g(\theta) = \cos \theta$ are periodic, of period 2π .

$$\sin(\theta + 2\pi) = \sin \theta$$

$$\cos(\theta + 2\pi) = \cos \theta$$

EXAMPLE 4.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations

(a) $2 \sin x - 1 = 0$

$$2 \sin x = 1 \quad (\div 2)$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{6} + 2n\pi$$

$$x = \frac{5\pi}{6} \quad x = \frac{5\pi}{6} + 2n\pi$$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{7\pi}{6}$	$-\frac{1}{2}$	0
$\frac{3\pi}{2}$	-1	0
2π	0	1

(b) $\cos^2 x - 3 \cos x + 2 = 0$

$$(\cos x - 2)(\cos x - 1) = 0$$

$$\cos x - 2 = 0$$

$$\cos x = 2$$

مرفوض
 $\cos x \in [-1, 1]$

$$\cos x - 1 = 0$$

$$\cos x = 1$$

$$x = 0, x = 2\pi$$

$$x = 2n\pi,$$

٠٥٩٠٦٤٧١٧١ $-1 \leq \cos \leq 1$

ن البر ١، ٢، ٣

Fundamental Trigonometric Identities

صنطا بقات متلينة

Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$$

$$\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

THEOREM 4.2

For any real numbers α and β , the following identities hold:

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

EXAMPLE 4.4 Deriving New Trigonometric Identities

Derive the identities

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\begin{aligned} \sin(\theta + \theta) &= \\ &= \sin \theta \cos \theta + \cos \theta \sin \theta \\ &= 2 \sin \theta \cos \theta \end{aligned}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\begin{aligned} \cos(\theta + \theta) &= \\ &= \cos \theta \cos \theta - \sin \theta \sin \theta \\ &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

0.4 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

○ The Inverse Trigonometric Functions

We thus define the inverse sine function by

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

EXAMPLE 4.5 Evaluating the Inverse Sine Function

Evaluate (a) $\sin^{-1} \left(\frac{\sqrt{3}}{2} \right)$

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \Rightarrow \sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{3}, \quad \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{3}$$

(b) $\sin^{-1} \left(-\frac{1}{2} \right)$.

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \Rightarrow \sin \theta = -\frac{1}{2}$$

$$\theta = -\frac{\pi}{6}, \quad -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\sin^{-1} \left(-\frac{1}{2} \right) = -\frac{\pi}{6}$$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{7\pi}{6}$	-1	0
2π	0	1

we define the inverse cosine function by

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \text{ and } 0 \leq y \leq \pi.$$

$$\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

$$\cos^{-1}(\cos x) = x, \quad \text{for all } x \in [0, \pi].$$

EXAMPLE 4.6 Evaluating the Inverse Cosine Function

Evaluate (a) $\cos^{-1}(0)$

$$\cos \theta = 0, \quad \theta \in [0, \pi]$$

$$\theta = \frac{\pi}{2} \in [0, \pi]$$

$$\cos^{-1} 0 = \frac{\pi}{2}$$

(b) $\cos^{-1}(-\frac{\sqrt{2}}{2})$.

$$\cos \theta = -\frac{\sqrt{2}}{2}, \quad \theta \in [0, \pi]$$

$$\theta = \frac{3\pi}{4} \in [0, \pi]$$

$$\cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

we define the inverse tangent function by

$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

EXAMPLE 4.7 Evaluating an Inverse Tangent

Evaluate $\tan^{-1}(1)$.

$$\tan \theta = 1, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow \theta = \frac{\pi}{4}, \quad \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

we define the inverse secant function by

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \quad \text{and} \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

EXAMPLE 4.8 Evaluating an Inverse Secant

Evaluate $\sec^{-1}(-\sqrt{2})$.

$$\sec \theta = -\sqrt{2}$$

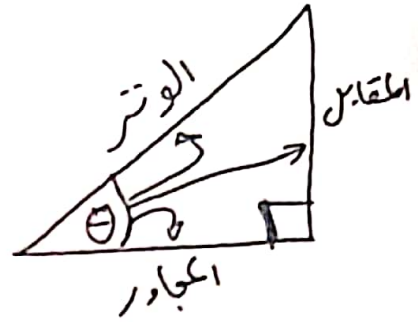
$$\cos \theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \theta = \frac{3\pi}{4} \in \left(\frac{\pi}{2}, \pi\right)$$

$$\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$$

$$\sin \theta = \frac{\text{المقابل}}{\text{الوتر}}$$

$$\cos \theta = \frac{\text{الجوار}}{\text{الوتر}}$$

$$\tan \theta = \frac{\text{المقابل}}{\text{الجوار}}$$



EXAMPLE 4.10 Simplifying Expressions Involving Inverse Trigonometric Functions

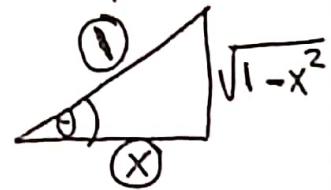
Simplify (a) $\sin(\cos^{-1} x)$

المطلوب $\sin(\theta)$

$$\theta = \cos^{-1} x$$

$$\cos \theta = \frac{x}{1} = \frac{\text{الجوار}}{\text{الوتر}}$$

$$= \frac{\text{المقابل}}{\text{الوتر}} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

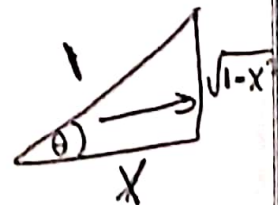


(b) $\tan(\cos^{-1} x)$

$$\tan \theta = \frac{\text{المقابل}}{\text{الجوار}} = \frac{\sqrt{1-x^2}}{x}$$

$$\theta = \cos^{-1} x$$

$$\cos \theta = \frac{x}{1} = \frac{\text{جوار}}{\text{وتر}}$$



EXERCISES 0.4



In exercises 1 and 2, convert the given radians measure to degrees.

$$\frac{180}{\pi} \text{ ضرب في}$$

1. (a) $\frac{\pi}{4}$

$$\frac{\pi}{4} \times \frac{180}{\pi} = \frac{180}{4} = 45^\circ$$

(c) $\frac{\pi}{6}$

$$\frac{\pi}{6} \times \frac{180}{\pi} = \frac{180}{6} = 30^\circ$$

(b) $\frac{\pi}{3}$

$$\frac{\pi}{3} \times \frac{180}{\pi} = \frac{180}{3} = 60^\circ$$

(d) $\frac{4\pi}{3}$

$$\frac{4\pi}{3} \times \frac{180}{\pi} = 4 \times 60 = 240^\circ$$

In exercises 3 and 4, convert the given degrees measure to radians.

$$\frac{\pi}{180} \text{ ضرب في}$$

3. (a) 180°

$$180^\circ \times \frac{\pi}{180} = \pi$$

(c) 120°

$$\frac{2}{3} \times \frac{\pi}{1} = \frac{2\pi}{3}$$

(b) 270°

$$\frac{3}{2} \times \frac{\pi}{1} = \frac{3\pi}{2}$$

(d) 30°

$$\frac{1}{6} \times \frac{\pi}{1} = \frac{\pi}{6}$$

In exercises 5–14, find all solutions of the given equation.

5. $2 \cos x - 1 = 0$

$$2 \cos x = 1 \quad \div 2$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} / x = -\frac{\pi}{3}$$

$$x = \frac{\pi}{3} + 2k\pi$$

$$x = -\frac{\pi}{3} + 2k\pi$$

8. $2 \sin x - \sqrt{3} = 0$

$$2 \sin x = \sqrt{3}$$

$$\sin x = \frac{\sqrt{3}}{2} \quad \div 2$$

$$x = \frac{\pi}{3} = 60^\circ$$

$$x = \frac{\pi}{3} + 2\pi$$

$$x = \frac{\pi}{3} + 2k\pi$$

In exercises 33-36, prove that the given trigonometric identity is true.

33. $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$

$$\sin(\alpha - \beta) = \sin(\alpha - (-\beta))$$

$$\sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta)$$

$$\sin \alpha \cos \beta \neq \cos \alpha \sin \beta \quad \left| \begin{array}{l} \text{even} \\ \cos(-\beta) = \cos \beta \\ \sin(-\beta) = -\sin \beta \\ \text{odd} \end{array} \right.$$

35. (a) $\cos(2\theta) = 2\cos^2 \theta - 1$

$$\cos(\theta + \theta) =$$

$$\cos \theta \cos \theta - \sin \theta \sin \theta$$

$$= \cos^2 \theta - \sin^2 \theta$$

$$= \cos^2 \theta - (1 - \cos^2 \theta)$$

$$= \cos^2 \theta + \cos^2 \theta - 1$$

$$= 2\cos^2 \theta - 1$$

(b) $\cos(2\theta) = 1 - 2\sin^2 \theta$

$$\cos(\theta + \theta) =$$

$$\cos \theta \cos \theta - \sin \theta \sin \theta$$

$$= \cos^2 \theta - \sin^2 \theta$$

$$= (1 - \sin^2 \theta) - \sin^2 \theta$$

$$= 1 - \sin^2 \theta - \sin^2 \theta$$

$$= 1 - 2\sin^2 \theta$$

In exercises 37-46, evaluate the inverse function by sketching a unit circle, locating the correct angle and evaluating the ordered pair on the circle.

37. $\cos^{-1} 0$

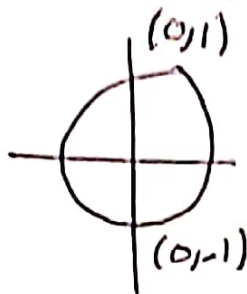
$$\cos \theta = 0$$

$$\theta = \frac{\pi}{2}$$

$$x = \cos \theta = 0$$

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1$$

$$y = \pm 1, (0, 1), (0, -1)$$



38. $\tan^{-1} 0$

$$\tan \theta = 0, \frac{\sin \theta}{\cos \theta} = 0$$

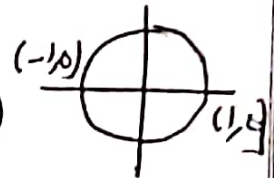
$$\sin \theta = 0$$

$$y = 0 \Rightarrow x^2 + y^2 = 1$$

$$x^2 = 1 \Rightarrow x = \pm 1$$

$$\cos \theta = \pm 1$$

$$(1, 0), (-1, 0)$$



40. $\cos^{-1}(1)$

$$\begin{matrix} \cos & \sin \\ (x, y) \end{matrix}$$

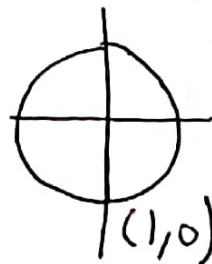
$$\cos \theta = 1$$

$$x = 1$$

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - 1 = 0$$

$$y = 0 \Rightarrow (1, 0)$$

$$\theta = 0$$



46. $\tan^{-1} \sqrt{3}$

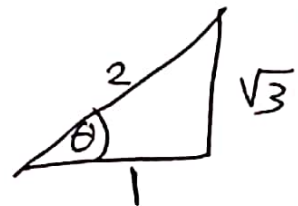
$$\tan \theta = \sqrt{3}, \theta = \frac{\pi}{3}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{3}}{1}$$

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\cos \theta = \frac{1}{2}$$

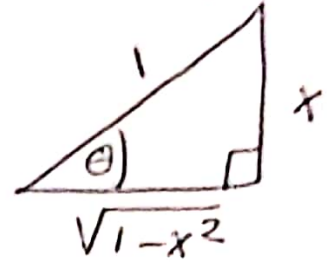
$$\left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$



In exercises 57–64, use a triangle to simplify each expression. Where applicable, state the range of x 's for which the simplification holds.

57. $\cos(\sin^{-1} x) \Rightarrow \theta = \sin^{-1} x \Rightarrow \sin \theta = x$

$$\cos \theta = \frac{\text{جوار}}{\text{وتر}} = \frac{\sqrt{1-x^2}}{1}$$

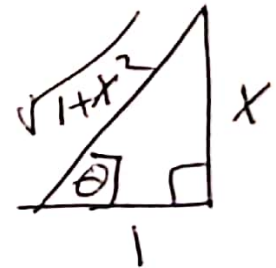


58. $\cos(\tan^{-1} x)$

$\Rightarrow \theta = \tan^{-1} x \Rightarrow \tan \theta = x$

$$\cos \theta = \frac{\text{جوار}}{\text{وتر}}$$

$$\cos \theta = \frac{1}{\sqrt{1+x^2}}$$



0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an exponential function because the variable t is in the exponent.

It's a simple matter to interpret fractional powers as roots. For instance,

$$x^{1/2} = \sqrt[2]{x^1}$$

$$x^{1/3} = \sqrt[3]{x^1}$$

$$x^{2/3} = \sqrt[3]{x^2}$$

$$\sqrt[m]{x^n} = x^{\frac{n}{m}}$$

RULES OF EXPONENTS (FOR $x, y > 0$)

For any real numbers p and q ,

$$\textcircled{1} x^p \cdot x^q = x^{p+q} \rightarrow x^2 \cdot x^3 = x^5$$

$$\textcircled{4} (xy)^p = x^p \cdot y^p \quad (2x)^2 = 2^2 x^2 = 4x^2$$

$$\textcircled{2} \frac{x^p}{x^q} = x^{p-q} \rightarrow \frac{x^5}{x} = x^4$$

$$\textcircled{5} \left(\frac{x}{y}\right)^p = \frac{x^p}{y^p}$$

$$\textcircled{3} (x^p)^q = x^{p \cdot q} \quad (x^2)^3 = x^6$$

$$\textcircled{6} x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

$$\textcircled{7} \frac{x^{-p}}{1} = \frac{1}{x^p}, \quad x^{-2} = \frac{1}{x^2} \quad / \quad 2^{-3} = \frac{1}{2^3}$$

EXAMPLE 5.1 Converting Expressions to Exponential Form

Convert each to exponential form:

(a) $3\sqrt[5]{x^5}$
 $= 3 X^{\frac{5}{2}}$

(b) $\frac{5}{\sqrt[3]{x^3}}$
 $= \frac{5}{X^{\frac{1}{3}}} = 5 X^{-\frac{1}{3}}$

(d) $(2^x \cdot 2^{3+x})^2$
 $= (2^{3+x+x})^2 = (2^{3+2x})^2$
 $= 2^{6+4x}$

(c) $\frac{3x^2}{2\sqrt[3]{x}}$
 $= \frac{3 X^2}{2 X^{\frac{1}{3}}} = \frac{3}{2} X^{2-\frac{1}{3}} = \frac{3}{2} X^{\frac{5}{3}}$

DEFINITION 5.2

For any positive number $b \neq 1$, the logarithm function with base b , written $\log_b x$, is defined by

$y = \log_b x$ if and only if $x = b^y$.

$\log_2 4 = 2$
 $2^2 = 4$

That is, the logarithm $\log_b x$ gives the exponent to which you must raise the base b to get the given number x . For example,

$\log_{10} 10 = 1$ (since $10^1 = 10$),
 $\log_{10} 100 = 2$ (since $10^2 = 100$),
 $\log_{10} 1000 = 3$ (since $10^3 = 1000$)

$\log_{10} 10$

$\log_a a = 1$

$\log_2 2 = 1$, $\log_{10} 10 = 1$

EXAMPLE 5.4 Evaluating Logarithms

Without using your calculator, determine

$$\log_{10} \frac{1}{10} = \frac{\log_{10} 10^{-1}}{10} = -1$$

$$\log(1/10) = -1$$

$$\frac{1}{10} = 10^{-1}$$

$$\log(0.001) = -3$$

$$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$$

$$\ln e = 1$$

$$\ln e \Rightarrow \log_e e = 1$$

$$\ln e^3$$

$$\ln e^3 = 3 \ln e = 3$$

In particular, for the base e, we have

$$\ln x = \log_e x \text{ for any } x > 0 \quad \text{and} \quad \ln(e^x) = x \text{ for any } x.$$

$$y = \ln x = \log_e x.$$

$$x = e^y = e^{\ln x}.$$

$$\ln e$$

EXAMPLE 5.5 Solving a Logarithmic EquationSolve the equation $\ln(x+5) = 3$ for x .

$$\ln(x+5) = 3 \Rightarrow e^{\ln(x+5)} = e^3$$

$$x+5 = e^3 \Rightarrow x = e^3 - 5$$

EXAMPLE 5.6 Solving an Exponential EquationSolve the equation $e^{x+4} = 7$ for x .

$$e^{x+4} = 7 \Rightarrow \ln e^{x+4} = \ln 7$$

$$x+4 = \ln 7 \Rightarrow \boxed{x = \ln 7 - 4}$$

THEOREM 5.1

$$\log 1 = 0$$

For any positive base $b \neq 1$,

- (i) $\log_b x$ is defined only for $x > 0$.
- (ii) $\log_b 1 = 0$ and
- (iii) if $b > 1$, then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

THEOREM 5.2For any positive base $b \neq 1$ and positive numbers x and y , we have

- (i) $\log_b(xy) = \log_b x + \log_b y$.
- (ii) $\log_b(x/y) = \log_b x - \log_b y$ and
- (iii) $\log_b(x^y) = y \log_b x$.

$$\log_2 2^2 = 2 \log_2 2 = 2$$

EXAMPLE 5.8 Simplifying Logarithmic ExpressionsWrite each as a single logarithm:

$$\left(\frac{1}{2}\right)^{-1} = 2$$

$$(a) \log_2 27^x - \log_2 3^x$$

$$\begin{aligned} \log_2 \frac{27^x}{3^x} &= \log_2 \frac{(3^3)^x}{3^x} \\ &= \log_2 \frac{3^{3x}}{3^x} = \log_2 3^{3x-x} \\ &= \log_2 3^{2x} = 2x \log_2 3 \end{aligned}$$

$$(b) \ln 8 - 3 \ln(1/2)$$

$$\begin{aligned} \ln 2^3 - 3 \ln 2^{-1} \\ \ln 2^3 + 3 \ln 2 = \ln 2^3 + \ln 2^3 \\ = \ln 2^6 = 6 \ln 2 \end{aligned}$$

EXAMPLE 5.9 Expanding a Logarithmic Expression

Use the rules of logarithms to expand the expression $\ln \left(\frac{x^3 y^4}{z^5} \right)$.

$$\begin{aligned}
 &= \ln(x^3 y^4) - \ln z^5 \\
 &= \ln x^3 + \ln y^4 - \ln z^5 \\
 &= 3 \ln x + 4 \ln y - 5 \ln z
 \end{aligned}$$

$$\begin{aligned}
 &\ln a^b \\
 &= b \ln a
 \end{aligned}$$

Ex: write 2^x , 5^x and $\left(\frac{2}{5}\right)^x$ as exponent of base e sol

$$(2^x) = e^{\ln 2^x} = e^{x \ln 2}$$

$$5^x = e^{\ln 5^x} = e^{x \ln 5}$$

$$\left(\frac{2}{5}\right)^x = e^{\ln \left(\frac{2}{5}\right)^x} = e^{x \ln \frac{2}{5}}$$

EXERCISES 0.5

المعادلات

In exercises 27-36, solve the given equation for x.

27. $e^{2x} = 2$ ← الد

$\ln e^{2x} = \ln 2$

$2x = \ln 2$

$x = \frac{\ln 2}{2}$

32. $x^2 \ln x - 9 \ln x = 0$

$(x^2 - 9) \ln x = 0$

$x^2 - 9 = 0 \quad | \quad \ln x = 0$
 $x^2 = 9 \quad | \quad x = 1$
 $x = \pm 3$

29. $e^{x^2-1} = 0$

الد

$e^x (x^2 - 1) = 0$

مستحيل = 0

$x^2 - 1 = 0$

$x^2 = 1 \rightarrow x = \pm 1$

35. $e^x = 1 + 6e^{-x}$

افضل

$e^x \cdot e^x = 1 \cdot e^x + 6(e^{-x} \cdot e^x)$

$e^{2x} = e^x + 6$

$e^{2x} - e^x - 6 = 0$

$(e^x + 2)(e^x - 3) = 0$

$e^x + 2 = 0 \quad | \quad e^x - 3 = 0$

$e^x = -2 \quad | \quad e^x = 3$

$x = \ln 3$

In exercises 37 and 38, use the definition of logarithm to determine the value.

38. (a) $\log_4 \frac{1}{16} = \log_4 \frac{1}{4^2} = \log_4 4^{-2} = -2 \log_4 4 = -2$

(b) $\log_4 2 = \log_4 4^{\frac{1}{2}} = \frac{1}{2} \log_4 4 = \frac{1}{2} \log_4 4 = \frac{1}{2}$

(c) $\log_9 3 = \log_9 9^{\frac{1}{2}} = \frac{1}{2} \log_9 9 = \frac{1}{2}$

وام

In exercises 41-46, rewrite the expression as a single logarithm.

42. $2 \ln 4 - \ln 3$

$\ln 4^2 - \ln 3$

$\ln \frac{4^2}{3} = \ln \frac{16}{3}$

46. $\ln 9 - 2 \ln 3$

$\ln 9 - \ln 3^2$

$\ln 9 - \ln 9 = \ln \frac{9}{9}$

$\ln 1 = 0$

In exercises 47-50, find a function of the form $f(x) = ae^{bx}$ with the given function values.

$e^0 = 1$

47. $f(0) = 2, f(2) = 6$

$f(0) = 2$

$ae^0 = 2$

$a = 2$

$6 = 2e^{2b}$

$\div 2$

$3 = e^{2b}$

$\ln$

$\ln 3 = \ln e^{2b}$

$\ln 3 = 2b \Rightarrow b = \frac{\ln 3}{2}$

$f(x) = 2e^{(\frac{\ln 3}{2})x} = 2(e^{\ln 3})^{\frac{x}{2}}$

$f(x) = 2(3)^{\frac{x}{2}}$